A parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$ for the subcritical case (Mathematical Analysis in Fluid and Gas Dynamics)

AUTHOR(S):

Nagai, Toshitaka

数理解析研究所講究録
第 1782 巻 2012 年 158-175

A parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$ for the subcritical case

Toshitaka Nagai
Department of Mathematics, Graduate School of Science, Hiroshima University,
Higashi-Hiroshima, 739-8526, JAPAN
e-mail: nagai@math.sci.hiroshima-u.ac.jp

Abstract

We consider the Cauchy problem of a parabolic-elliptic system in $\mathbb{R}^2$, which is a mathematical model of chemotaxis. We review the application of rearrangements to the Cauchy problem with subcritical mass, that is, the total mass is less than 8π.

2000 Mathematical Subject Classification: 35B45, 35K15, 35K55
Keywords : chemotaxis system, subcritical mass, rearrangement techniques

1 Introduction

In this paper we consider the Cauchy problem for the following nonlinear parabolic equation with a non-local term in $\mathbb{R}^2$:

$$\text{(CP)} \qquad \begin{cases} \partial_t u = \Delta u - \nabla \cdot (u(\nabla N * u)), & t > 0, x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2. \end{cases}$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, $N = N(x)$ is the Newtonian potential in $\mathbb{R}^2$ and $\nabla N * u$ is the convolution of ∇N and u with respect to the space variable, namely

$$N(x) = \frac{1}{2\pi} \log \frac{1}{|x|}, \qquad (\nabla N * u)(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} u(t,y)\, dy.$$

The Cauchy problem (CP) comes from the following Cauchy problem for a parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$:

$$(\mathrm{CP})_\psi \qquad \begin{cases} \partial_t u = \Delta u - \nabla \cdot (u\nabla\psi), & t>0, x\in\mathbb{R}^2, \\ -\Delta\psi = u, & t>0, x\in\mathbb{R}^2, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^2. \end{cases}$$

Since the Poisson equation admits solutions up to constants, we specify ψ as

$$\psi(t,x) = (N*u)(t,x) = \int_{\mathbb{R}^2} N(x-y)u(t,y)\,dy.$$

This system is a simplified version of chemotaxis model derived from the original parabolic system due to Keller-Segel [25] (see also Childress-Percus [15]). In the chemotaxis model, $u \geq 0$ denotes the density of microorganisms and ψ the concentration of a chemical-attractant secreted by themselves. The system is also a model of self-attracting particles in $\mathbb{R}^2$ (see [10, 43]), where u is the density of particles in $\mathbb{R}^2$ interacting with themselves through the potential ψ.

One of the basic properties of nonnegative solutions to (CP) is the conservation of the total mass, namely

$$\int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx, \quad t>0,$$

and the global existence and large-time behavior of nonnegative solutions to (CP) heavily depend on the total mass. In fact, in the subcritical case $\int_{\mathbb{R}^2} u_0(x)\,dx < 8\pi$, the nonnegative solution to (CP) exists globally in time (see [13, 33]), and converges to a radially symmetric self-similar solution (see [9, 13, 34]). On the other hand, in the supercritical case $\int_{\mathbb{R}^2} u_0(x)\,dx > 8\pi$, the nonnegative solution may blow up in finite time (see [10, 13, 26]). In the critical case $\int_{\mathbb{R}^2} u_0(x)\,dx = 8\pi$, at least three types of solutions appear: a solution tending to $8\pi\delta_{x_0}$ as time goes to infinity (see [12, 38]), where δ_{x_0} is the Dirac delta function at x_0 and x_0 is the center of mass of u_0, a solution tending to a stationary solution (see [9, 11]), and an oscillating solution in time (see [35]). Related results for (CP) as a chemotaxis model, for example see [7, 8, 21, 23, 28, 29, 32, 36, 39, 40], and as models of self-attracting particles, see [6, 7], and the references cited therein. We also refer [22, 41] in which we can find related results for chemotaxis models.

For the subcritical case, in [13] they have studied the global existence of nonnegative weak solutions to $(\mathrm{CP})_\psi$ for the nonnegative initial data u_0 satisfying

$$(1.1) \qquad u_0,\ u_0\log u_0,\ |x|^2u_0 \in L^1(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} u_0(x)\,dx < 8\pi.$$

Their main tools are the free energy inequality

$$F[u(t)] + \int_0^t \int_{\mathbb{R}^2} u|\nabla \log u - \nabla\psi|^2 \,dxds \le F[u_0], \quad t > 0,$$

where $F[u]$ is the free energy given by

$$F[u] = \int_{\mathbb{R}^2} u \log u \,dx - \frac{1}{2}\int_{\mathbb{R}^2} u\psi \,dx,$$

the second moment identity

$$\int_{\mathbb{R}^2} u(t)|x|^2 \,dx = \int_{\mathbb{R}^2} u_0|x|^2 \,dx + 4M\Big(1 - \frac{M}{8\pi}\Big)t$$

and the logarithmic Hardy-Littlewood-Sobolev inequality (see Lemma 2.4 in [13]). Hence assumption (1.1) on the initial data u_0 is essentially needed in their proof. We remark that the uniqueness of weak solutions to $(\mathrm{CP})_\psi$ seems to be open.

In this paper we review the application of rearrangements to the Cauchy problem (CP) for the subcritical case $M := \int_{\mathbb{R}^2} u_0(x)\,dx < 8\pi$ based on the results in [33, 34]. Rearrangement techniques are useful to get isoperimetric inequalities for elliptic equations and parabolic equations, which give the estimates on the L^p-norms of solutions for these equations (see [1, 2, 3, 16, 30, 31, 37, 42] for example). We first apply rearrangement techniques to get the global existence and decay estimates of nonnegative mild solutions to (CP) (see Theorem 4.1) under the following assumption on the nonnegative initial data u_0:

$$u_0 \in L^1(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} u_0(x)\,dx < 8\pi. \tag{1.2}$$

We estimate the L^p-norms of the nonnegative solution u to (CP) by comparing the L^p-norms between the solution u and a radially symmetric self-similar solution U_M. In the subcritical case, given $0 < M < 8\pi$, we have a radially symmetric self-similar solution U_M of (CP) such that

$$U_M(t,x) = \frac{1}{t}\Psi\left(\frac{|x|}{\sqrt{t}}\right), \quad \int_{\mathbb{R}^2} U_M(t,x)\,dx = M, \tag{1.3}$$

where Ψ is positive, integrable and bounded on $[0,\infty)$. The existence of such a radially symmetric self-similar solution has been studied in [5] by ODE methods and in [36] by PDE methods, and uniqueness in [9]. The result reads as follows: For given $M \in (0, 8\pi)$, there exists uniquely a radially symmetric self-similar solution U_M satisfying (1.3). If U_M exists, then $M \in (0, 8\pi)$.

We next mention the uniqueness of nonnegative weak solutions to (CP) with initial data $M\delta_0$, where δ_0 is the Dirac delta function at th origin. By rearrangement techniques, we have $v = U_M$ for the nonnegative weak solution v with initial data $M\delta_0$, where $0 < M < \pi$ (see Theorem 5.1).

Throughout this paper, we use the following notation: $L^p(\mathbb{R}^d)$ is the Lebesgue space on $\mathbb{R}^d$ with the usual norm $\|\cdot\|_{L^p}$ for $1 \leq p \leq \infty$. In the case $d = 2$, for simplicity, we denote $L^p(\mathbb{R}^2)$ and $\|\cdot\|_{L^p}$ by L^p and $\|\cdot\|_p$, respectively. For $Q \subset \mathbb{R}^d$ and a Banach space X, we denote the set of all continuous functions from Q to X by $C(Q; X)$ and the set of all bounded continuous functions by $BC(Q; X)$. If $X = \mathbb{R}$, then we denote $C(Q; \mathbb{R})$ and $BC(Q; \mathbb{R})$ by $C(Q)$ and $BC(Q)$, respectively. Denote by $\mathbb{Z}_+$ the set of all nonnegative integers. For $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{Z}_+^d$, put $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and

$$\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}, \quad \partial_j = \frac{\partial}{\partial x_j}.$$

For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, we denote by ∂_x^m any partial derivative of order m with respect to the space variables and put

$$\|\partial_x^m f\|_{L^p} = \sum_{|\alpha|=m} \|\partial_x^\alpha f\|_{L^p}.$$

For a function $f = f(t, x), (t, x) \in (a, b) \times \Omega$, where $-\infty \leq a < b \leq \infty, \Omega \subset \mathbb{R}^d$, we denote by $f(t) : \Omega \to \mathbb{R}$ for $t \in (a, b)$ the function $f(t)(x) = f(t, x)$.

This paper is organized as follows. Section 2 is devoted to the local existence, uniqueness and regularity of mild solutions to (CP). In Section 3 we mention some properties of decreasing rearrangements. In Section 4 we review the global existence and decay estimates of nonnegative solutions to (CP) only under assumptions (1.2), and in Section 5 the uniqueness of nonnegative weak solutions with initial data $M\delta_0 (0 < M < 8\pi)$. In Section 6 we give remarks to a parabolic-elliptic system replacing the second equation in $(\mathrm{CP})_\psi$ by $-\Delta\psi + \psi = u$.

2 Local existence of solutions in time

We begin with the definition of mild solutions to the Cauchy problem (CP).

Definition 2.1. Given $u_0 \in L^1$, a function u on $[0, T) \times \mathbb{R}^2$ is said to be a mild solution of (CP) on $[0, T)$ if

(i) $u \in C([0,T); L^1) \cap C((0,T); L^{4/3})$,

(ii) $\sup_{0<t<T} t^{1/4}\|u(t)\|_{4/3} < \infty$,

(iii) u satisfies the integral equation

$$(2.1) \quad u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s))\,ds, \quad 0 < t < T,$$

where $e^{t\Delta}$ is the heat semigroup defined by

$$(e^{t\Delta}f)(x) = \int_{\mathbb{R}^2} G(t, x-y)f(y)\,dy, \quad G(t,x) = \frac{1}{4\pi t}\exp(-\frac{|x|^2}{4t}).$$

A function u on $[0,\infty) \times \mathbb{R}^2$ is a global mild solution of (CP) with initial data u_0 if u is a mild solution of (CP) on $[0,T)$ for any $0 < T < \infty$.

Remark. The integral in (2.1) is well-defined by (i) and (ii) of Definition 2.1, applying the well-known $L^p - L^q$ estimates for the heat semigroup $e^{t\Delta}$ in $\mathbb{R}^2$

$$\|\partial_t^m \partial_x^n e^{t\Delta} f\|_p \le C t^{-1/q+1/p-m-n/2}\|f\|_q, \quad f \in L^q,$$

where $1 \le q \le p \le \infty$ and m and n are nonnegative integers, and the following inequality: For $4/3 \le q < 2$,

$$(2.2) \qquad \|f(\nabla N * g)\|_{2q/(4-q)} \le C_q \|f\|_q \|g\|_q \quad \text{for all} \quad f, g \in L^q,$$

where C_q is a positive constant depending only on q. Inequality (2.2) is obtained from the Hardy-Littlewood-Sobolev inequality in $\mathbb{R}^2$: For $1 < q < 2$,

$$\|\frac{1}{|x|} * g\|_{2q/(2-q)} \le C_q \|g\|_q \quad \text{for all} \quad g \in L^q,$$

where C_q is a positive constant depending only on q.

To mention local existence, uniqueness and regularity, following Kato [24], we introduce function spaces. Let $T > 0$. For $1 \le p \le \infty$ and $\gamma \ge 0$, define the Banach space $C_{\gamma,T}(L^p)$ with norm $\|\cdot\|_{p,\gamma,T}$ by

$$C_{\gamma,T}(L^p) = \{u \mid u \in C((0,T); L^p), \ \sup_{0<t<T} t^\gamma \|u(t)\|_p < \infty\},$$
$$\|u\|_{p,\gamma,T} = \sup_{0<t<T} t^\gamma \|u(t)\|_p \quad \text{for } u \in C_{\gamma,T}(L^p).$$

For $\gamma > 0$, define $\dot{C}_{\gamma,T}(L^p)$ by

$$\dot{C}_{\gamma,T}(L^p) = \{u \in C_{\gamma,T}(L^p) \mid \lim_{t\to 0} t^\gamma \|u(t)\|_p = 0\},$$

and for $\gamma = 0$, $\dot{C}_{0,T}(L^p) = BC([0,T); L^p)$. $\dot{C}_{\gamma,T}(L^p)$ is a closed subspace of $C_{\alpha,T}(L^p)$.

The local existence, uniqueness and regularity of mild solutions to (CP) was obtained by methods similar to those for the vorticity equation in $\mathbb{R}^2$ in [4, 14, 20, 24]. For the proof of Proposition 2.1, see [33].

Proposition 2.1. *Given $u_0 \in L^1$, there exists $T \in (0,\infty)$ such that the Cauchy problem* (CP) *corresponding to the initial data u_0 has uniquely a mild solution u on $[0,T)$. Moreover, u satisfies the following:*

(i) $u(t) \to u_0$ *in* L^1 *as* $t \to 0$.

(ii) *For* $1 \le q \le \infty$, $u \in \dot{C}_{1-1/q,T}(L^q)$.

(iii) *For* $\ell \in \mathbb{Z}_+, \alpha \in \mathbb{Z}_+^2$ *and* $1 < q < \infty$, $\partial_t^\ell \partial_x^\alpha u \in \dot{C}_{1-1/q+|\alpha|/2+\ell,T}(L^q)$.

(iv) *Let* $\ell \in \mathbb{Z}_+, \alpha \in \mathbb{Z}_+^2$. *For* $2 < q < \infty$ *if* $|\alpha| = 0$, *and for* $1 < q < \infty$ *if* $|\alpha| \ge 1$,
$$\partial_t^\ell \partial_x^\alpha (\nabla N * u) \in \dot{C}_{1/2-1/q+|\alpha|/2+\ell,T}(L^q).$$

(v) u *is a classical solution of* $\partial_t u = \Delta u - \nabla \cdot (u(\nabla N * u))$ *in* $(0,T) \times \mathbb{R}^2$.

(vi) $\int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx$ *for* $0 < t < T$.

(vii) *If* $u_0 \log(1+|x|) \in L^1$, *then* $u(t)\log(1+|x|) \in L^1$ *for* $0 < t < T$.

(viii) *If* $u_0 \ge 0, u_0 \not\equiv 0$ *on* $\mathbb{R}^2$, *then* $u(t,x) > 0$ *on* $(0,T) \times \mathbb{R}^2$.

Remark 2.1. By Proposition 2.1, for $u_0 \in L^1$ satisfying $u_0 \log(1+|x|) \in L^1$, the Cauchy problem $(\mathrm{CP})_\psi$ has uniquely a mild solution u, because $\psi(t) = N * u(t)$ is well-defined in L^1_{loc} by $u(t)\log(1+|x|) \in L^1$, and $\nabla\psi = \nabla N * u$ and $-\Delta\psi = u$ are satisfied.

We characterize the maximal existence time of solutions in terms of the modified entropy $\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx$.

Proposition 2.2. *Let T_m be the maximal existence time of u. If $T_m < \infty$, then*
$$\limsup_{t\to T_m} \int_{\mathbb{R}^2} (1+u(t))\log(1+u(t))\,dx = +\infty.$$

For the proof of this proposition, see [33]. This proposition implies that if the following a priori estimate
$$\int_{\mathbb{R}^2} (1+u(t))\log(1+u(t))\,dx \le C_T, \quad T/2 \le t \le T$$
holds for any $0 < T < T_m$, where C_T is a constant depending on $T \in (0,\infty)$, then the solution u exists globally in time.

3 Decreasing rearrangements

For a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ and $\theta \in \mathbb{R}$, we use the following notation for simplicity:

$$\{f > \theta\} := \{x \in \mathbb{R}^d : \ f(x) > \theta\}, \quad |f > \theta| := |\{x \in \mathbb{R}^d : \ f(x) > \theta\}|,$$

where $|A|$ is the Lebesgue measure of a Lebesgue measurable set A in $\mathbb{R}^d$.

For a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we assume that f vanishes at infinity in the sense that $||f| > \theta| < \infty$ for all $\theta > 0$. We define the distribution function μ_f of f by

$$\mu_f(\theta) = ||f| > \theta| \quad (\theta \geq 0),$$

and the decreasing rearrangement f^* of f, the generalized inverse of μ_f, by

$$f^*(s) = \inf\{\theta \geq 0 : \ \mu_f(\theta) \leq s\} \quad (s \geq 0).$$

We also define the function $f^\sharp : \mathbb{R}^d \to \mathbb{R}$, called the symmetric rearrangement or the Schwarz symmetrization of f, by

$$f^\sharp(x) = f^*(c_d|x|^d),$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$.

We refer to [3, 27, 30, 37] for the basic properties of rearrangements mentioned below and for Proposition 3.1.

(i) f^* is non-increasing and right-continuous on $[0, \infty)$.

(ii) $f^*(0) = \|f\|_{L^\infty(\mathbb{R}^d)}, \quad f^*(\infty) = 0$.

(iii) If f is continuous on $\mathbb{R}^d$, then f^* and $f^\sharp$ are continuous on $[0, \infty)$ and $\mathbb{R}^d$, respectively.

Proposition 3.1. (i) *For every Borel measurable function Φ from $\mathbb{R}$ to $[0, \infty)$,*

$$\int_{\mathbb{R}^d} \Phi(|f(x)|)\, dx = \int_{\mathbb{R}^d} \Phi(f^\sharp(x))\, dx = \int_0^\infty \Phi(f^*(s))\, ds.$$

(ii) *Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be integrable on $\mathbb{R}^d$. If $\int_0^s f^*(\sigma)\, d\sigma \leq \int_0^s g^*(\sigma)\, d\sigma$ for all $s > 0$, then*

$$\int_{\mathbb{R}^d} \Phi(|f(x)|)\, dx \leq \int_{\mathbb{R}^d} \Phi(|g(x)|)\, dx$$

for all convex functions $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$.

(iii) (The Hardy-Littlewood inequality) *Let* $1 \le p, q \le \infty, 1/p + 1/q = 1$. *For* $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |f(x)||g(x)|\,dx \le \int_{\mathbb{R}^d} f^\sharp(x) g^\sharp(x)\,dx = \int_0^\infty f^*(s) g^*(s)\,ds.$$

(iv) (Contraction property) *Let* $1 \le p \le \infty$. *For* $f, g \in L^p(\mathbb{R}^d)$,

$$\|f^* - g^*\|_{L^p(0,\infty)} = \|f^\sharp - g^\sharp\|_{L^p(\mathbb{R}^d)} \le \|f - g\|_{L^p(\mathbb{R}^d)}.$$

(v) (The Pólya-Szegö inequality) *Let* $1 \le p \le \infty$. *If* $f \in W^{1,p}(\mathbb{R}^d)$, *then* $f^\sharp \in W^{1,p}(\mathbb{R}^d)$ *and*

$$\|\nabla f^\sharp\|_{L^p(\mathbb{R}^d)} \le \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

Let $v = v(t, x)$ be a smooth function on $(0, T) \times \mathbb{R}^2$ such that $v(t)$ is in $L^1 \cap L^\infty$ and radially symmetric in x for every $0 < t < T$, and v satisfies

$$\partial_t v = \Delta v - \nabla \cdot (v(\nabla N * v)) \quad \text{in} \quad (0, T) \times \mathbb{R}^2,$$

where

$$(\nabla N * v)(t, x) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} v(t, y)\,dy.$$

Define $\varphi(t, s)$ by $v(t, x) = \varphi(t, s)$, $s = \pi|x|^2$. Then the following hold(see Lemma 5.1 of [33]):

(i) φ satisfies

$$\partial_t \varphi(t, s) = 4\pi \partial_s (s \partial_s \varphi(t, s)) + \partial_s \left(\varphi(t, s) \int_0^s \varphi(t, \sigma)\,d\sigma \right).$$

(ii) $\Phi(t, s) := \int_0^s \varphi(t, \sigma)\,d\sigma$ satisfies

$$\partial_t \Phi(t, s) = 4\pi s \partial_s^2 \Phi(t, s) + \Phi(t, s) \partial_s \Phi(t, s). \tag{3.1}$$

Let u be a nonnegative mild solution of (CP) on $[0, T)$ with nonnegative initial data $u_0 \in L^1$. For the decreasing rearrangement u^* of the solution u with respect to the space variable x, define the function $H(t, s)$ by

$$H(t, s) = \int_0^s u^*(t, \sigma)\,d\sigma, \quad 0 < t < T, \ s \ge 0.$$

By the regularity of u (see Proposition 2.1) and the Pólya-Szegö inequality in Proposition 3.1, we have the following.

Proposition 3.2. *Let $1 < p < \infty$. It hold that*

(i) $H(t,0) = 0,\ H(t,\infty) = \int_{\mathbb{R}^2} u_0(x)\,dx,\ 0 < t < T$,

(ii) $H \in BC([0,T) \times [0,\infty))$ *and* $H(0,s) = \int_0^s u_0^*(\sigma)\,d\sigma, s > 0$,

(iii) $\partial_s H \in BC((T_0,T) \times (0,\infty)) \cap L^\infty(0,T;L^1(0,\infty))$ *for any* $0 < T_0 < T$,

(iv) $\partial_s^2 H \in L^\infty(T_0,T;L^p(s_0,\infty))$ *for any* $0 < T_0 < T, s_0 > 0$,

(v) $\partial_t H \in L^\infty(T_0,T;L^p(0,R))$ *for any* $0 < T_0 < T, R > 0$.

The function H satisfies the following differential inequality (3.2) in Proposition 3.3, which is a key one to get the L^p-estimates of u. For the proof, see [17, 18, 33].

Proposition 3.3. *For almost all* $t \in (0,T)$,

$$\partial_t H - 4\pi s \partial_s^2 H - H \partial_s H \leq 0, \quad a.a.\ s > 0. \tag{3.2}$$

4 Global existence and decay estimates of non-negative solutions

We first remark that the equation of u in (CP)

$$\partial_t u = \Delta u - \nabla \cdot (u(\nabla N * u)), \quad t > 0,\ x \in \mathbb{R}^2 \tag{4.1}$$

has a scaling invariant property such that for a solution u of (4.1), the function u_λ for $\lambda > 0$ defined by

$$u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x), \quad t > 0,\ x \in \mathbb{R}^2$$

is also a solution of (4.1). If $u_\lambda = u$ for all $\lambda > 0$, the solution u is called a self-similar solution.

As mentioned in the introduction, given $M \in (0, 8\pi)$ there exists uniquely a radially symmetric self-similar solution U_M of (CP) satisfying (1.3). We introduce the mass distribution function

$$\tilde{M}(t,s) = \int_{|x| \leq \sqrt{s}} U_M(t,x)\,dx, \quad t > 0,\ s \geq 0$$

and see that $\tilde{M}(t,s)$ satisfies the following:

$$\begin{cases} \partial_t \tilde{M} = 4\partial_s^2 \tilde{M} + \dfrac{1}{\pi} \tilde{M} \partial_s \tilde{M}, & t > 0,\ s > 0, \\ \tilde{M}(t,0) = 0,\ \tilde{M}(t,+\infty) = M, & t > 0, \\ \lim\limits_{t \to 0} \tilde{M}(t,s) = M, & s > 0. \end{cases}$$

Since it is satisfied that for each $\lambda > 0$,

$$\tilde{M}(\lambda t, \lambda s) = \tilde{M}(t,s), \quad t > 0,\ s \geq 0,$$

$\tilde{M}(t,s)$ has the form

$$\tilde{M}(t,s) = m\Big(\frac{s}{t}\Big), \quad t > 0,\ 0 \leq s < \infty.$$

The nonnegative function $m(y)$ satisfies

$$\begin{cases} 4\dfrac{d^2m}{dy^2}(y) + \dfrac{dm}{dy}(y) + \dfrac{1}{\pi y} m(y)\dfrac{dm}{dy}(y) = 0, \ y > 0, \\ m(0) = 0, \quad m(+\infty) = M, \end{cases}$$

and it was shown in Lemma 4.1 of [9] that

$$\tag{4.2} \begin{cases} m \in C^1([0,\infty)), \ \dfrac{dm}{dy}(y) > 0, \ \dfrac{d^2m}{dy^2}(y) < 0, \ y > 0, \\ M(1 - e^{-y/4}) \leq m(y) \leq \min\left\{4\dfrac{dm}{dy}(0)(1 - e^{-y/4}), M\right\}, \quad y > 0, \\ \dfrac{dm}{dy}(y) \leq \dfrac{dm}{dy}(0)e^{-y/4}, \quad y > 0. \end{cases}$$

Observing

$$\tag{4.3} U_M(t,x) = \frac{1}{\pi}\partial_s \tilde{M}(t,|x|^2) = \frac{1}{\pi t}\frac{dm}{dy}\left(\frac{|x|^2}{t}\right),$$

the radially symmetric function $U_M(t,x)$ is decreasing with respect to $|x|$, and hence

$$\tag{4.4} U_M(t,x) = U_M^\sharp(t,x) = U_M^*(t, \pi|x|^2),$$

where $U_M^\sharp$ and U_M^* are the Schwarz symmetrization and the decreasing rearrangement of U_M with respect to the space variable x, respectively. By (4.2) and (4.3), for $1 \leq p \leq \infty$,

$$\tag{4.5} \|U_M(t)\|_p \leq C_{M,p} t^{-1+1/p}, \quad t > 0,$$

where $C_{M,p}$ is a positive constant depending only on M and p.

Define $V(t,s)$ by

$$V(t,s) = \int_0^s U_M^*(t,\sigma)\, d\sigma, \quad t > 0,\ s \geq 0.$$

From (4.3) and (4.4) it is easily seen that $U_M^*(t,s) = (\pi t)^{-1} dm/dy((\pi t)^{-1}s)$ and

$$V(t,s) = m\left(\frac{s}{\pi t}\right).$$

In view of (3.1) and $\int_0^\infty U_M^*(t,\sigma)\,d\sigma = \int_{\mathbb{R}^2} U_M(t,x)\,dx = M$, we see that $V(t,s)$ satisfies

$$\begin{cases} \partial_t V - 4\pi s \partial_s^2 V - V\partial_s V = 0, & t>0,\ s>0, \\ V(t,0)=0,\ V(t,\infty)=M, & t>0, \\ \lim\limits_{t\to 0+} V(t,s) = M, & s>0. \end{cases}$$

For a nonnegative initial data $u_0 \in L^1$ satisfying $M := \int_{\mathbb{R}^2} u_0\,dx < 8\pi$, let u be the nonnegative mild solution of (CP) on $[0,T)$. Then by Proposition 3.3, the function $H(t,s) = \int_0^s u^*(t,\sigma)\,d\sigma$ satisfies that for almost all $t \in (0,T)$,

$$\partial_t H - 4\pi s\partial_s^2 H - H\partial_s H \le 0, \quad \text{a.a. } s>0.$$

At $s=0$ and $s=+\infty$,

$$H(t,0)=0, \quad H(t,+\infty) = \int_0^\infty u^*(t,\sigma)\,d\sigma = \int_{\mathbb{R}^2} u(t,x)\,dx = M.$$

At the initial time $t=0$,

$$\lim_{t\to 0+} (H(t,s) - V(t,s)) = \int_0^s u_0^*(\sigma)\,d\sigma - M \le 0 \quad (s>0).$$

Hence, by calculations similar to those in [33], we obtain

$$H(t,s) \le V(t,s), \quad t>0,\ s>0.$$

Therefore we have the following.

Proposition 4.1 (Proposition 5.3, [33]). *It holds that for each* $0<t<T$,

$$\int_0^s u^*(t,\sigma)\,d\sigma \le \int_0^s U_M^*(t,\sigma)\,d\sigma \quad \textit{for all } s>0.$$

As an application of Proposition 4.1, we have the following.

Theorem 4.1 (Theorem 5.1, [33]). *For the nonnegative initial data* $u_0 \in L^1$ *satisfying*

$$M := \int_{\mathbb{R}^2} u_0\,dx < 8\pi,$$

let u be the nonnegative mild solution u of (CP) *on* $[0,T)$. *Then it hold that for each* $0<t<T$,

$$\|(1+u(t))\log(1+u(t))\|_1 \le \|(1+U_M(t))\log(1+U_M(t))\|_1, \tag{4.6}$$

$$\|u(t)\|_p \le \|U_M(t)\|_p \quad \textit{for all} \ \ 1\le p\le\infty. \tag{4.7}$$

Hence the solution u exists globally in time and for every $1<p\le\infty$ *the decay estimates*

$$\|u(t)\|_p \le C_{M,p}t^{-1+1/p} \quad \textit{for} \ \ t>0 \tag{4.8}$$

hold, where $C_{M,p}$ is a positive constant depending only on M and p.

Proof. Take convex functions Φ from $[0,\infty)$ to $[0,\infty)$ as follows:

$$\Phi(v)=(1+v)\log(1+v), \quad \Phi(v)=v^p \ (1\le p<\infty).$$

Then (4.6) and (4.7) with $1\le p<\infty$ follow from Proposition 4.1 and (ii) in Proposition 3.1. Letting $p\to\infty$ in (4.7) with $1\le p<\infty$, we obtain (4.7) for $p=\infty$.

Global existence follows from (4.6) and Proposition 2.2. The decay estimates (4.8) follow from (4.7) and (4.5). □

5 Uniqueness of weak solutions with delta functions as initial data

In this section we discuss the uniqueness of weak solutions of (CP) with initial data $M\delta_0$, where $0<M<8\pi$ and δ_0 is the Dirac delta function at the origin. For this purpose, we begin with the definition of weak solutions.

Definition 5.1. A function v on $(0,\infty)\times\mathbb{R}^2$ is said to be a weak solution of (CP) with initial data $M\delta_0$, where $M\in\mathbb{R}$, if

(i) $v\in C((0,\infty);L^1\cap L^{4/3})$,

(ii) $\sup_{0<t<1} t^{1/4}\|v(t)\|_{4/3}<\infty$,

(iii) for any $\varphi\in C_0^\infty([0,\infty)\times\mathbb{R}^2)$, v satisfies

$$0=M\varphi(0,0)+\int_0^\infty\int_{\mathbb{R}^2}(\partial_t\varphi+\Delta\varphi)v\,dxdt+\int_0^\infty\int_{\mathbb{R}^2}\nabla\varphi\cdot(v(\nabla N*v))\,dxdt.$$

The following theorem implies the uniqueness of nonnegative weak solutions of (CP) with initial data $M\delta_0$, where $0 < M < 8\pi$.

Theorem 5.1 (Theorem 4.1, [34]). *Let v be a nonnegative weak solution of* (CP) *with initial data $M\delta_0$. If $0 < M < 8\pi$, then $v = U_M$.*

Remark 5.1. Uniqueness seems to be still open for $M \geq 8\pi$, and generally, for weak solutions with finite measure as initial data.

The proof of Theorem 5.1 relies on the following proposition.

Proposition 5.1 (Proposition 4.2, [19]). *Let $f, g : \mathbb{R}^d \to [0, +\infty)$ be continuous and integrable functions satisfying*

(i) $\int_0^s f^*(\sigma)\, d\sigma \leq \int_0^s g^*(\sigma)\, d\sigma$ *for all $s > 0$,*

(ii) *g is radially symmetric and non-increasing with respect to $|x|$,*

(iii) $\int_{\mathbb{R}^d} f(x)\, dx = \int_{\mathbb{R}^d} g(x)\, dx$,

(iv) $\int_{\mathbb{R}^d} |x|^d f(x)\, dx = \int_{\mathbb{R}^d} |x|^d g(x)\, dx < \infty$.

Then $f = g$.

In what follows, we give the outline of the proof of Theorem 5.1. By Definition 5.1, we first observe that for the nonnegative weak solution v of (CP) with the initial data $M\delta_0$, it hold that for every $0 < t < T$,

$$\int_{\mathbb{R}^2} v(t, x)\, dx = M,$$
$$\int_{\mathbb{R}^2} |x|^2 v(t, x)\, dx = 4M\Big(1 - \frac{M}{8\pi}\Big)t.$$

Since $\int_{\mathbb{R}^2} U_M(t, x)\, dx = M$ and U_M is also a nonnegative weak solution of (CP) with initial data $M\delta_0$, we have

$$\int_{\mathbb{R}^2} |x|^2 U_M(t, x)\, dx = 4M\Big(1 - \frac{M}{8\pi}\Big)t.$$

Hence, for every $0 < t < T$,

$$\int_{\mathbb{R}^2} v(t, x)\, dx = \int_{\mathbb{R}^2} U_M(t, x)\, dx (= M), \tag{5.1}$$

$$\int_{\mathbb{R}^2} |x|^2 v(t, x)\, dx = \int_{\mathbb{R}^2} |x|^2 U_M(t, x)\, dx. \tag{5.2}$$

We claim that for every $t > 0$,

$$\int_0^s v^*(t,\sigma)\,d\sigma \le \int_0^s U_M^*(t,\sigma)\,d\sigma \quad \text{for all} \quad s > 0. \tag{5.3}$$

Indeed, for an arbitrary number $\tau > 0$ being fixed, define the function $w(t,x) = v(t+\tau,x)$ on $[0,\infty) \times \mathbb{R}^2$. Then we see that w is in $C([0,\infty); L^1 \cap L^{4/3})$ and a nonnegative mild solution of (CP) corresponding to the initial data $v(\tau) \in L^1 \cap L^{4/3}$. Since $\int_{\mathbb{R}^2} v(\tau)\,dx = M < 8\pi$, applying Proposition 4.1 yields that for each $t > 0$,

$$\int_0^s v^*(t+\tau,\sigma)\,d\sigma = \int_0^s w^*(t,\sigma)\,d\sigma \le \int_0^s U_M^*(t,\sigma)\,d\sigma \quad \text{for all} \quad s > 0. \tag{5.4}$$

We observe $\|v^*(t+\tau) - v^*(t)\|_1 \to 0$ as $\tau \to 0$ by the contraction property of the decreasing rearrangement, and hence, letting $\tau \to 0$ in (5.4), we conclude (5.3).

Now we can apply Proposition 5.1 as $f = v(t)$ and $g(t) = U_M(t)$ $(t > 0)$ by virtue of (5.1), (5.2) and (5.3), and obtain $v(t) = U_M(t)$ for every $t > 0$. Thus we establish Theorem 5.1.

6 Remarks on another parabolic-elliptic stem of drift-diffusion type

Consider the following Cauchy problem for a parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$:

$$(\mathrm{KS})_\psi \qquad \begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \psi), & t > 0, x \in \mathbb{R}^2, \\ -\Delta \psi + \psi = u, & t > 0, x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2. \end{cases}$$

The difference between $(\mathrm{KS})_\psi$ and $(\mathrm{CP})_\psi$ is only the equation on ψ. For a nonnegative initial data $u_0 \in L^1$, let (u,ψ) be a nonnegative solution of $(\mathrm{KS})_\psi$ on the time interval $[0,T)$, and define the function

$$H(t,s) = \int_0^s u^*(t,\sigma)\,d\sigma, \quad 0 < t < T, \; s \ge 0,$$

where u^* is the decreasing rearrangement of u with respect to x. Similar to the way the differential inequality (3.2) is derived, we obtain that for almost all $t \in (0,T)$,

$$\partial_t H - 4\pi s \partial_s^2 H - H \partial_s H \le 0, \quad a.a.\ s > 0.$$

Assuming $M := \int_{\mathbb{R}^2} u_0\, dx < 8\pi$, by the similar procedure to that in Section 4 we deduce that for all $t \in (0,T)$,

$$\int_0^s u^*(t,\sigma)\, d\sigma \le \int_0^s U_M^*(t,\sigma)\, d\sigma \quad \text{for all} \quad s > 0,$$
$$\|u(t)\|_p \le \|U_M(t)\|_p \quad \text{for all } 1 \le p \le \infty,$$

where U_M is the radially symmetric self-similar solution of (CP) satisfying (1.3). Hence the solution (u, ψ) exists globally in time and the following decay estimate

$$\|u(t)\|_p \le Ct^{-1+1/p}, \quad t > 0$$

holds for every $1 \le p \le \infty$, where C is a positive constant depending on M and p.

References

[1] A. Alvino, P.-L. Lions, and G. Trombetti, Comparison results for elliptic and parabolic equations via symmetrization: a new approach, Differential Integral Equations, **4**(1991), 25–50.

[2] C. Bandle, On symmetrization in parabolic equations, J. d'Anal. Math., **30** (1976), 98–112.

[3] C. Bandle, Isoperimetric Inequalities and Applications, Pitman, London, 1980.

[4] M. Ben-Artzi, Global solutions of two-dimensional Navier-Stokes and Euler equations, Arch. Rational Mech. Anal., **128** (1994), 329–358.

[5] P. Biler, Growth and accretion of mass in an astrophysical model, Applicationes Mathematicae (Warsaw), **23** (1995), 179–189.

[6] P. Biler, Existence and nonexistence of solutions for a model of gravitational interaction of particles, III, Colloq. Math., **68** (1995), 229–239.

[7] P. Biler, M. Cannone, I. A. Guerra, and G. Karch, Global regular and singular solutions for a model of gravitational particles, Math. Ann., **330** (2004), 693–708.

[8] P. Biler, D. Hilhorst, and T. Nadzieja, Existence and nonexistence of solutions for a model of gravitational interaction of particles, II, Colloq. Math., **67** (1994), 297–308.

[9] P. Biler, G. Karch, P. Laurençot, and T. Nadzieja, The 8π−problem for radially symmetric solutions of a chemotaxis model in the plane, Math. Meth. Appl. Sci., **29** (2006), 1563–1583.

[10] P. Biler and T. Nadzieja, Existence and nonexistence of solutions for a model of gravitational interactions of particles, I, Colloq. Math., **66** (1994), 319–334.

[11] A. Blanchet, E. Carlen, and J. A. Carrillo, Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model, arXiv:1009.0134.

[12] A. Blanchet, J. A. Carrillo, and N. Masmoudi, Infinite time aggregation for the critical Paltak-Keller-Segel model in $\mathbb{R}^2$, Comm. Pure Appl. Math., **61** (2008), 1449–1481.

[13] A. Blanchet, J. Dolbeault, and B. Perthame, Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions, Electron. J. Differential Equations, **2006** (2006) No. 44, 1–33.

[14] H. Brezis, Remarks on the preceding paper by M. Ben-Artzi: Global solutions of two-dimensional Navier-Stokes and Euler equations, Arch. Rational Mech. Anal., **128** (1994), 359–360.

[15] S. Childress and J. K. Percus, Nonlinear aspects of chemotaxis, Math. Biosci., **56** (1981), 217–237.

[16] J. I. Diaz, Nonlinear Partial Differential Equations and Free Boundaries, Vol 1. Elliptic Equations, Research Notes in Math. Vol. 106, Pitman, London, 1985.

[17] J. I. Diaz and T. Nagai, Symmetrization in a parabolic-elliptic system related to chemotaxis, Adv. Math. Sci. Appl., **5** (1995), 659–680.

[18] J. I. Diaz, T. Nagai, and J. M. Rakotoson, Symmetrization techniques on unbounded domains: Application to a chemotaxis system in $\mathbb{R}^N$, J. Differential Equations, **145** (1998), 156–183.

[19] I. Gallagher, T. Gallay, and P.-L. Lions, On the uniqueness of the solution of the two-dimensional Navier-Stokes equation with a Dirac mass as initial vorticity, Math. Nachr., **278** (2005), 1665–1672.

[20] Y. Giga, T. Miyakawa, and H. Osada, Two-dimensional Navier-Stokes flow with measures as initial vorticity, Arch. Rational Mech. Anal., **96** (1988), 223–250.

[21] M. A. Herrero and J. J. L. Velázquez, Singularity patterns in a chemotaxis model, Math. Ann., **306** (1996), 583-623.

[22] D. Horstmann, From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I, Jahresber. Deutsch. Math.-Verein., **105** (2003), 103–165.

[23] W. Jäger and S. Luckhaus, On explosions of solutions to a system of partial differential equations modeling chemotaxis, Trans. Amer. Math. Soc., **329** (1992), 819-824.

[24] T. Kato, The Navier-Stokes equation for an incompressible fluid in R^2 with a measure as the initial vorticity, Differential Integral Equations, **7** (1994), No 4, 949–966.

[25] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., **26** (1970), 399-415.

[26] M. Kurokiba and T. Ogawa, Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type, Differential Integral Equations, **16** (2003), 427-452.

[27] E. H. Lieb and M. Loss, Analysis, Graduate Studies in Mathematics, 14, Ameri. Math. Soc., Providence, RI, 2001.

[28] S. Luckhaus and Y. Sugiyama, Asymptotic profile with the optimal convergence rate for a parabolic equation of chemotaxis in super-critical cases, Indiana Univ. Math. J., 56 (2007), 1279–1297

[29] N. Mizoguchi and T. Senba, Type II blowup solutions to a simplified chemotaxis system, Adv. Differential Equations, **17** (2007), 505–545.

[30] J. Mossino, Inégalités Isopérimétriques et Applications en Physique, Hermann, Paris, 1984.

[31] J. Mossino and J. M. Rakotoson, Isoperimetric inequalities in parabolic equations, Ann. Scuola Norm. Sup. Pisa, **13** (1986), 51–73.

[32] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl., **5** (1995), 581-601.

[33] T. Nagai, Global existence and decay estimates of solutions to a parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$, Differential Integral Equations, **24** (2011), 29–68.

[34] T. Nagai, Convergence to self-similar solutions for a parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$, Adv. Differential Equations, **16** (2011), 839–866.

[35] Y. Naito and T. Senba, Bounded and unbounded oscillating solutions to a parabolic-elliptic system in two dimensional solutions, preprint.

[36] Y. Naito and T. Suzuki, Self-similar solutions to a nonlinear parabolic-elliptic system, Proceedings of Third East Asia Partial Differential Equation Conference, Taiwanese J. Math., **8** (2004), 43–55.

[37] J. M. Rakotoson, Réarrangement Relatif: un instrument d'estimation dans les problèmes aux limites, Springer-Verlag, Berlin, 2008.

[38] T. Senba, Grow-up rate of a radial solution for a parabolic-elliptic system in R^2, Adv. Differential Equations, **14** (2009), 1155–1192.

[39] T. Senba and T. Suzuki, Chemotactic collapse in a parabolic-elliptic system of mathematical biology, Adv. Differential Equations, **6** (2001), 21-50.

[40] Y. Sugiyama, Global existence in subcritical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems, Differential Integral Equations, **19** (2006), 841–876.

[41] T. Suzuki, Free Energy and Self-Interacting Particles, Progress in Nonlinear Differential Equations and their Applications, 62, Birkhäuser Boston Inc., Boston, MA, 2005.

[42] G. Talenti, Elliptic equations and rearrangements, Ann. Sc. Norm. Pisa, **3** (1976), 679–718.

[43] G. Wolansky, Comparison between two models of self-gravitating clusters: conditions for gravitational collapse, Nonlinear Anal., **24** (1995), 1119–1129.